MULTI-RESPONSE OPTIMIZATION ON INTEGRATED LOCATION AND DISPERSION EFFECTS

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ABSTRACT

With the advance of response surface optimization, an extended approach of integrated average and standard deviation of multiple responses is addressed in this paper. In addition to the location effect of the process performance, the dispersion effect should also be taken into account with the development of products. The multi-response optimization problem is discussed under a specific paradigm of non-linear programming (NLP). The combination of the mean square error and standard deviation is used to construct the objective of the mathematical model. Meanwhile, the scaling issue in multi-response optimization is resolved through the standardization of the numerical data. An NLP approach is proposed to seek the compromised optimum of the mathematical model. The illustrative examples are used to verify the proposed approach and compare to the existing methods in the literature.

KEY WORDS: Response surface methodology (RSM); Non-linear programming (NLP); Newton method; Steepest descent method.

INTRODUCTION

Most of processes are traditionally associated with the optimization of the single response in the early work. With the complexity and variety of process development, manufactured products are typically concerned with various quality characteristics. In practice, there are several responses of interest in most experiments. To achieve the optimal settings simultaneously, the exact adjustment of input variables needs to be considered thoroughly in the industrial process. This type of procedure involves choosing a set of input factors that will result in a product with the best combination of the quality characteristics. In general, the process is referred to as multi-response design optimization with the quality characteristics defined as the multiple responses (denoted by \( y_1, y_2, \ldots, y_K \)). The primary scheme is to demonstrate a set of input factors that provide the “trade-off” solutions of the multiple responses. Recently, the major approaches of the multi-response optimization problems focus on the location effect merely without considering the dispersion effect of the responses. In consequence, all quality characteristics are assumed to perform equal variation. In fact, it always possesses significant difference among responses in practice. In such a case, the current multi-response optimal approaches can not clarify and work out effectively.

Form the past attempts, it have considered only one property in making any adjustment without taking into account. This was mainly due to the lack of a systematic way of simultaneously considering all the responses, even though the balance among all properties defines the overall quality. A manufactured product is often evaluated by several characteristics. Optimizing the manufacturing process purely with respect to any single-response variable will, indeed, lead to non-optimum values for the remaining characteristics. It is desirable to find an overall optimum or a best compromise of the product characteristics simultaneously. Several quantitative methods have been developed, which combine the multiple responses into a single function and seek to find the optimal compromise. Each of the characteristics of the product is described by a response variable. Controllable design variables are considered the input factors to the manufacturing process. The goal of multi-response optimization is to find the settings of the design variables that attain an optimal compromise of the response variables. By optimal compromise, it indicates that finding the operating level of the design variables such that each product characteristic is as “close” as possible to its ideal value.

Response surface methodology (RSM) originated from Box and Wilson (1951) is an integrated tool for process and product optimization. It contains a set of mathematical techniques, which consists of statistical designed experiments, regression analysis, and elementary optimization. In the context of RSM, the quality engineer (or referred to as “experimenter”) intends to locate the optimal setting for the input factors that results in a product with the best combination of multiple quality characteristics. In common RSM practice, designed experiments are often used to investigate the performance of a process and system. It can be illustrated through the generic engineering system as shown in Figure 1. The whole process is being treated as a “black box,” where the internal process dynamics are intentionally ignored. Instead, the functional relationship will be examined between some of the controllable factors, \( x_1, x_2, \ldots, x_i \), and a specific set
of the response variables of interest, \( y_1, y_2, \ldots, y_m \). The uncontrollable factors, \( z_1, z_2, \ldots, z_q \), can be deemed the noise variables the quality engineer fails to take into account.

LITERATURE REVIEW FOR THE MULTI-RESPONSE OPTIMIZATION

We will revisit briefly several important multi-response optimization procedures existing in the RSM literature, however, all of which are designed for the purpose of off-line process improvement. As in a single-response experiment, one of the objectives is the determination of optimum conditions on the input variables that optimize the predicted responses. However, the definition of an optimum in a multi-response situation is more complex than in the single-response problem. The reason for this is that when more than one response variable are considered simultaneously, the meaning of an optimum becomes unclear for the decision of a multi-response function. A most common approach to solving multi-response optimization problem is a unifying objectives approach; that is, the individual responses are mathematically combined to form a single objective function. Unifying objective approaches are used in the quality area to optimize several responses all together. Initially, the individual responses are modeled by creating a response surface from a designed experiment. The set of response surface is then subjected to a mathematical transformation which acts as a normalizing agent so that all responses can then be combined into a single function. Consequently, by varying the levels of the controllable factors, an optimal objective function and hence optimal controllable factor setting can be obtained. In an effort to solving the multi-response optimization problem, some researchers resorted purely to the superimposition of response contours to arrive at overlapped operating conditions (see, e.g., Lind, Goldin and Hickman 1960). Contour plots provide a pictorial description of the behavior of the multi-response system. This method, although simple and straightforward, has its limitations in large systems involving several controllable factors and several responses. The method of ridge analysis, first introduced by Hoerl (1959, 1964) and later refined by Draper (1963) for the optimization of a single (univariate) response function that could be well modeled with a second-order response surface model, has no analog in a multi-response situation. Myers and Carter (1973) developed a similar method for the optimization of a primary response function, subject to the condition that a constraint response function takes on some desirable target value.

Harrington (1965) introduced an analytic technique for the optimization of a multi-response function based on the concept of utility or desirability of a property associated with a given response function. By incorporating the individual desirability values into a single overall desirability value; viz., their geometric mean, Harrington proposed using exponential-type transformations on each response so as to obtain a measure of the overall quality of the system. The multivariate optimization problem is then reduced to the univariate maximization of the overall desirability function. Later, Derringer and Suich (1980) extended Harrington’s procedure by introducing more general transformations of the responses into desirability values. The desirability function approach to the problem of multi-response optimization is clear-cut, easy to apply, and permits the user to give subjective judgments on the importance of each response. Nevertheless, due to the subjective nature of this approach, experience on the part of the user in assessing a product’s desirability value is necessary to achieve accurate results. In the face of the practical problems, a typical situation is considered more than two responses, leading to simultaneous optimization. Usually, operating conditions that result in the individual response optimized mete out unsatisfactory performance in other responses. Among the conflicting responses, the approach has to search the “compromised” optimum which conforms to the practitioner’s preference. In addition to the desirability function method, a variety of approaches have been proposed and suggested, including the generalized distance function (Khuri and Conlon 1981), Taguchi’s loss function (Pignatiello 1993), the variance-covariance approach (Vining 1998), and so on. A comparison of multi-response optimization methods was reported in Wurl and Albin (1999), and Kros and Mastrangelo (2001, 2004). These techniques are either applied in the practical problems or made a study of theory. While there are several multi-response techniques available, little has been said regarding the explicit instructions of how to use them on-line to optimize actual multi-response processes.

Khuri and Conlon (1981) introduced a multi-response optimization technique called the generalized distance approach. This distance metric uses squared deviations of the product characteristics from their targets, and then normalizes these deviations by the variance of prediction of the response variables. Later, Vining (1998) proposed a mean squared error method, which allows the practitioner to specify the directions of economic importance for the compromise optimum while making allowance for the variance-covariance structure of the
multiple responses explicitly. Kros and Mastrangelo (2004) investigated the relationship between different response types when they are mixed (i.e. NTB, LTB, or STB). The research outcome demonstrated that the mix of response types impacts greatly the choice of final input parameters and response levels achieved. Del Castillo (1996) presented a methodology for analyzing multi-responses processes by using confidence regions or confidence cones. Nonlinear programming (NLP) techniques are used to locate operating conditions that simultaneously satisfy confidence regions or cones of the responses under study. It is assumed that there are no linear dependencies among the responses. Kim and Lin (2000) suggested that an exponential desirability functional form simplified the desirability function assessment process. This approach presented does not require any assumptions according to the degree of the estimated response models, and henceforth is robust to the potential dependences between response variables. Later, Kim and Lin (2006) proposed an integrated approach to simultaneously optimizing both the location and dispersion effects of multiple responses. The proposed approach overcomes the common limitation of the existing multi-response ones, which typically ignore the dispersion effects of the responses.

NON-LINEAR PROGRAMMING APPROACH TO PERFORM THE PROCEDURE OF MULTI-RESPONSE PROCESS IMPROVEMENT

MODEL DEVELOPMENT AND SCALING SCHEME

Researchers have sought to combine Taguchi’s RD principles with conventional RSM in order to model the response directly as a function of control factors. RSM is a statistical tool that is useful for modeling and analysis in situations, where the response of interest is affected by several input factors. In addition, RSM is typically used to optimize this response by estimating an input-response functional form when the exact functional relationship is not known or is very complicated. RSM is often viewed in the context of experimental design, model fitting, and optimization. Let \( \hat{\mu}(x) \) and \( \hat{\sigma}(x) \) represent the fitted response functions for the mean and standard deviation of the control with \( k \) independent variables in the design space. The first-order fitted response functions for the mean and standard deviation are obtained as:

\[
\hat{\mu}(x) = \hat{a_0} + \hat{a_1}x_1 + \hat{a_2}x_2 + \cdots + \hat{a_k}x_k,
\]

\[
\hat{\sigma}(x) = \hat{b_0} + \hat{b_1}x_1 + \hat{b_2}x_2 + \cdots + \hat{b_k}x_k,
\]

where \( \hat{a_0} \) and \( \hat{b_0} \) are regression constants, \( a \) and \( b \) vectors with constant values. Different optimization approaches can now be applied by using different formulations for the objective function to be minimized. Using the regression models \( \hat{\mu}(x) \), \( \hat{\sigma}(x) \) or a combination of both in the objective function will yield different optimization results. The following paragraphs will examine in more detail various formulations for the objective function. The \( x_i \) terms are control factors, and the estimates of the \( a \)’s and \( b \)’s in the functions are linear regression coefficients of the first-order fitted responses for the process mean and standard deviation, respectively.

For the different quality characteristics, it might bring about the numerical and scaling issues. Consequently, the standardization of the raw data will proceed through mean and standard deviation. The standardized formula is shown as follows:

\[
\frac{y_i - \bar{y}}{s/\sqrt{n}}
\]

where the mathematical model is presented as the form

\[
\text{Minimize } \sum (\hat{\mu}_i - \hat{T}_x)^2 + (\hat{\sigma}_i - \hat{T}_x)^2 \quad \text{for } i = 1 \sim n
\]

\( s.t \)

\[-1 \leq x_j \leq 1 \quad \text{for } j = 1 \sim n\]

OPTIMIZATION APPROACH

Newton’s method is based on exploiting the quadratic approximation of the function \( \theta \) at a given point \( \lambda_k \). This quadratic approximation \( q \) is given by

\[
q(\lambda) = \theta(\lambda_k) + \theta'(\lambda_k)(\lambda - \lambda_k) + \frac{1}{2}\theta''(\lambda_k)(\lambda - \lambda_k)^2
\]

The point \( \lambda_{k+1} \) is taken to be the point where the derivative of \( q \) is equal to zero. This yields \( \theta'(\lambda_k) + \theta''(\lambda_k)(\lambda - \lambda_k) = 0 \), so that
\[ \lambda_{k+1} = \lambda_k - \frac{\theta'(\lambda_k)}{\theta''(\lambda_k)} \]  

(4)

The procedure is terminated when \( |\lambda_{k+1} - \lambda_k| < \varepsilon \), or when \( |\theta'(\lambda_k)| < \varepsilon \), where \( \varepsilon \) is pre-specified termination scalar.

The method of steepest ascent requires performing a sequence of sets of experimental trials. Each set is considered as a result of proceeding sequentially along a path of maximum (or minimum) increase in the values of a current response \( y \) observed in an experiment. The procedure of steepest ascent depends on approximating a response surface model with a hyperplane in some restricted region. The model is fitted into a first-order polynomial initially by running experimental trials. Then, the first-order approximate model is used to determine the steepest path. However, due to a possible curvature in the response surface, the initial increase in the response will be likely followed by making flat or even a decrease. By this step, a new series of experiments is performed and the resulting data are used to fit another first-order model. A new path is determined along which increasing response values may be observed. This process continues until it becomes evident that little or no additional increase in the response can be gained. Frequently, the initial estimate of the optimum operating conditions for the system will be far from the actual optimum. In such circumstances, the objective of the experimenter is to move rapidly near the neighborhood of the true (but unknown) optimum. In practice, it is expected to use a simple and economically efficient experimental procedure. When we are remote from the true optimum, we usually assume that a first-order model is an adequate approximation to the true surface in a small region of the \( x \)'s. The method of steepest ascent is an iterative procedure for moving sequentially along the path of steepest ascent; that is, in the direction of the maximum increase in the response. Of course, if minimization is desired, then we call this technique the method of steepest descent. Suppose that the fitted first-order model obtained is

\[ \hat{y} = \hat{\beta}_0 + \sum_{i=1}^k \hat{\beta}_i x_i \]  

(5)

and the first-order response surface, that is, the contour of \( \hat{y} \), is a series of parallel lines such as shown in figure 2. The direction of steepest ascent is the direction where \( \hat{y} \) increases most rapidly. This direction is parallel to the normal line to the fitted response surface.

Experiments are performed along the path of steepest ascent until no further increase in response is observed. Then, a new first-order model may be suitable, a new path of steepest ascent determined, and the procedure continued. Eventually, the experimenter will arrive in the vicinity of the optimum. At this stage, it is usual to indicate a lack of fit of a first-order model. Additional experiments should be augmented to the current set of experimental runs for obtaining a more precise estimate of the optimum.

![Figure 2. First order response surface and path of steepest ascent.](image)

Response surface analysis combines interdisciplinary optimization techniques that are particularly useful for the investigation of scientific problems, in which a response of interest is influenced by several controllable factors and the objective is to optimize the response. The major aspect of RSM is to help the process engineer determine the optimum setting of the controllable factors that result in desirable response outputs. Suppose that the process engineer is concerned with a product, process, or system involving a response \( y \) that relies on the (natural) controllable factors \( \xi_1, \xi_2, \ldots, \xi_k \). The relationship can be expressed as

\[ y = f(\xi_1, \xi_2, \ldots, \xi_k) + \varepsilon \]  

(6)

where the form of the true response function \( f \) is frequently unknown and perhaps very complicated, and \( \varepsilon \) is a term that represents other sources of variability unaccounted for in \( f \). For example, \( \varepsilon \) could include effects such as measurement error on the response, other sources
of variation inherent in the process or system (background noise, or common cause variation in the language of statistical process control), and so forth. We will treat $\epsilon$ as a statistical error, often assuming it to have a normal distribution with mean zero and known variance $\sigma^2$. If the mean of $\epsilon$ is zero, then the expectation of a response variable is

$$E(y) = \eta = E[f(\xi_1, \xi_2, \ldots, \xi_k)] + E(\epsilon) = f(\xi_1, \xi_2, \ldots, \xi_k).$$

(7)

The variables $\xi_1, \xi_2, \ldots, \xi_k$ are usually called the natural variables, because they are expressed in the natural units of measurement. In much RSM work, it is convenient to transform the natural variables to coded variables $x_1, x_2, \ldots, x_k$. The coded variables are usually defined to be dimensionless with mean zero and the same spread or standard deviation. In terms of the coded variables, the true response can be written as

$$\eta = f(x_1, x_2, \ldots, x_k).$$

(8)

Suppose that $f$ is a function defined on some interval of the real numbers. There always exists a function $g$ that imitates the behavior of $f$ on some interval. The $g$ function is an approximation to $f$. In general, some (application) problems may have difficult, complex calculation involving $f$, and $g$ could be calculated easily and approximates to $f$. In this study, it shall be considered by means of a polynomial approximation. By doing so, an estimated direction for multi-response optimization can possibly be obtained. Assuming that $f$ is a continuous, smooth function and twice differentiable at some point, express $x = x_0$. A second-order Taylor series expansion of $f$ constructed at the point $x = x_0$ is

$$f(x) = f(x_0) + \nabla f(x_0)^T (x - x_0) + \frac{1}{2} (x - x_0)^T H(x)(x - x_0)$$

(9)

where $\nabla f(x_0)$ is the gradient vector of $f$ at $x_0$ and $H(x)$ is the Hessian matrix of $f$ at $x_0$. Thus, the foregoing expansion can be reduced to a polynomial of the form:

$$f = \beta_0 + \sum_{i=1}^{k} \beta_i x_i,$$

(10)

the first-order response surface model if only the first two terms are considered, and

$$f = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_i x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j,$$

(11)

the second-order response surface model if the second-order effects are taken into account. These two models are the most frequently used approximations in the RSM practice. Of practical importance is the first-order model, the linear coefficient estimates of which are used as performing experiments along the path of steepest ascent for process improvement. The next section will illustrate how the method of steepest ascent method is extended to the multi-response case based on the non-linear programming approach.

Typically, the initial estimate of the optimum operating conditions for the system may be far from the real optimum. In such circumstances, the experimenter expects to move the experimental block rapidly to the vicinity of the true optimum. It is customary to use a simple and economic procedure for experimental optimization. Assume that a first-order model is an adequate approximation to the true surface in a local region of the controllable factor space. The method of steepest ascent is a procedure for moving sequentially along the path of steepest ascent; that is, in the direction of the maximum increase in the response. Of course, if minimization is desired, then the technique is called the method of steepest descent via pre-multiplying the direction of steepest ascent by -1. The fitted first-order model can be expressed by

$$\hat{y} = \hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i x_i,$$

(12)

the first-order response surface; that is, the contours of $\hat{y}$, is a series of parallel lines such as that shown in figure 2 for the case of $k = 2$. The direction of steepest ascent is the direction in which $\hat{y}$ increases most rapidly. This direction is parallel to the normal line to the contours of fitted response surface. In common RSM practice, it is usual to take the path of steepest ascent as the line through the center of the region of interest (experimental region) and normal to the fitted surface. Thus, the steps along the path are proportional to the estimated first-order regression coefficients $\hat{\beta}_i$. The actual step size is determined by the experimenter’s process knowledge or other practical considerations. Experimentation is sequentially conducted along the path of steepest ascent until no further improvement in response is observed. Then, a new first-order model may be fit. By means of collecting additional experimental runs around the current operating point, a new path of steepest ascent is determined and the procedure is continued. Ultimately, the experimenter will reach the vicinity of the true optimum, where a lack of fit indication of a first-order model should be present. Thus, additional experiments should be augmented for building a higher-order response surface model so as to obtain a more precise estimate of the optimum.

While a path created by moving $x_j$ a relative distance that is proportional to the regression coefficient $\hat{b}_j$, is a reasonable and intuitive selection, a better understanding of the procedure can be gained through a mathematical development of the procedure. Consider the
fitted first-order response surface model
\[ \hat{y} = b_0 + b_1x_1 + b_2x_2 + \cdots + b_kx_k. \]  

Along the path of steepest ascent, it indicates that which yields a maximum estimated response with the equality constraint where a radial experimental region is considered to fit the first-order response surface model. Namely, of all points that are a fixed distance \( r \) from the design center, we seek that \( x_1, x_2, \ldots, x_k \) for which \( \hat{y} \) is maximized. Note that in the metric of the coded design variables, the design center is the origin \((0, 0, \cdots, 0)\). As a result, the constraint given by \( \sum_{i=1}^{k} x_i^2 = r^2 \) is that of a sphere with radius \( r \). The solution to this optimization problem involves the use of Lagrange multipliers. Maximization requires the partial derivatives with respect to \( x_j \) \((j = 1, 2, \ldots, k)\) of the Lagrangian function
\[ L = b_0 + b_1x_1 + b_2x_2 + \cdots + b_kx_k - \lambda \left( \sum_{i=1}^{k} x_i^2 - r^2 \right). \]  

The partial derivative with respect to \( x_j \) is
\[ \frac{\partial L}{\partial x_j} = b_j - 2\lambda x_j \quad \text{for} \quad j = 1, 2, \ldots, k. \]  

Setting \( \frac{\partial L}{\partial x_j} = 0 \) gives the following coordinate of \( x_j \) of the path of steepest ascent, as follows
\[ x_j = \frac{b_j}{2\lambda}. \]  

Now, the quantity \( 1/2\lambda = \rho \) may be viewed as a scaled constant of proportionality. That is, the coordinates are given by
\[ x_1 = \rho b_1, x_2 = \rho b_2, \ldots, x_k = \rho b_k, \]  
\[ \rho = \frac{r}{\sqrt{\sum_{i=1}^{k} b_i^2}}. \]  

For steepest ascent, the constant \( \rho \) is positive and for steepest descent, it is taken to be negative. At the moment, it implies that the choice of \( \rho \), relating to \( \lambda \), merely determines the distance from the design center that the deciding point will dominate. Certainly, the constant \( \rho \), how far the step length should be placed in steepest ascent, is determined by the practitioner.

It is straightforward to give a general algorithm for determining the coordinates of a point on the path of steepest ascent. Assume that the origin point \( x_1 = x_2 = \cdots = x_k = 0 \) is the base or the starting point. Then

1. Choose a step size in one of the process variables, say \( \Delta x_i \). Usually, we select the variable we know the most about, or we select the variable that has the largest (or nearly the largest) absolute regression coefficient \( |b_i| \).
2. The step size in the other variables is calculated by \( \Delta x_j = \frac{b_j}{b_j/\Delta x_j} \), \( j = 1, 2, \ldots, k, \ i \neq j \).
3. Convert the \( \Delta x_j \)'s from the coded variables to the natural variables.

We perform the multi-response modeling and then apply the method of steepest ascent direction to the resulting combined function. However, the resultant function is, in essence, presented the nonlinear form. To facilitate process optimization on-line using the method of steepest ascent will be employed to a nonlinear combined function with mean and standard deviation. Subsequently, the method of steepest ascent aforementioned comes into play for on-line multi-response process optimization.

**ILLUSTRATIVE EXAMPLES AND DATA ANALYSIS**

A practical problem with multiple responses reported in the chemical engineering literature (Jaureg et al., 1997) is employed to demonstrate the use of the existing multi-response optimization approaches. When surfactant solutions are mixed at high-speed, micro bubbles (10–100 \( \mu \)m in diameter) are formed. It is postulated that these bubbles, called colloidal gas aphrons (CGAs), are composed of a gaseous inner core surrounded by a thin soapy film. The properties of the CGAs are measured by three different responses—stability \( y_1 \), volumetric ratio \( y_2 \), and temperature \( y_3 \). The responses, \( y_1, y_2 \) and \( y_3 \) are larger-the-better (LTB), smaller-the-better (STB), and nominal-the-best (NTB)-type responses, respectively. The purpose of the experiment was to determine the effects of concentration of surfactant \( x_1 \), concentration of salt \( x_2 \), and time of stirring \( x_3 \) on the CGA properties. The original experiment was conducted in a...
central composite design with eight factorial points, six axial points, and a center point. The center point was replicated six times and the other design points were replicated twice. In virtue of this replication by each level of factors, response surface models can be fitted with both the mean and the standard deviation of each response. The raw data is represented in Table 1. By considering the scaling issue and optimization approach, the primary data is transformed into the form of the full design with three factors. The new data is displayed in Table 2. The measurements in the table 3 are the bounds and targets of three responses.

Table 1. The CGA study data

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Table 2. The transformation data of three full factorial design

<table>
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<th>u</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>ȳ_u1</th>
<th>ȳ_u2</th>
<th>ȳ_u3</th>
<th>s_u1</th>
<th>s_u2</th>
<th>s_u3</th>
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</table>
Table 3. The constraint of three responses

<table>
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<tr>
<th>Bounds and target</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\mu_i^{\text{min}}, \mu_i^{\text{max}}, y_i^{\text{max}})$</td>
<td>(3.00, 7.00, 7.00)</td>
<td>(0.10, 0.10, 0.60)</td>
<td>(15.00, 30.00, 45.00)</td>
</tr>
<tr>
<td>$(\mu_i^{\text{min}}, \mu_i^{\text{max}}, y_i^{\text{max}})$</td>
<td>(0.00, 0.00, 0.10)</td>
<td>(0.00, 0.00, 0.10)</td>
<td>(1.00, 1.00, 2.00)</td>
</tr>
</tbody>
</table>

The linear regression of transformation data represents as it follows:

\[
\begin{align*}
T_{u_1} &= 0.088 - 0.386 x_1 - 2.205 x_2 - 0.195 x_3 \quad (18) \\
T_{u_2} &= -0.591 - 0.100 x_1 - 1.645 x_2 - 0.498 x_3 \quad (19) \\
T_{u_3} &= -0.857 - 0.095 x_1 + 0.856 x_2 - 0.449 x_3 \quad (20) \\
s_{u_1} &= 1.297 - 0.199 x_1 + 1.396 x_2 + 0.319 x_3 \quad (21) \\
s_{u_2} &= 0.281 - 0.342 x_1 + 0.569 x_2 + 1.936 x_3 \quad (22) \\
s_{u_3} &= 0.461 - 0.076 x_1 + 0.682 x_2 - 1.223 x_3 \quad (23)
\end{align*}
\]

Table 4. The optimal results of Newton method with the starting point $(x_1, x_2, x_3) = (1, 1, 1)$

<table>
<thead>
<tr>
<th>Factorial settings $(x_1, x_2, x_3)$</th>
<th>(0, -0.1960, -0.5741)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objective value</td>
<td>93.7868</td>
</tr>
</tbody>
</table>

Table 5. The optimal results of steepest ascent method with the starting point $(x_1, x_2, x_3) = (1, 1, 1)$

<table>
<thead>
<tr>
<th>Factorial settings $(x_1, x_2, x_3)$</th>
<th>(1.0, -0.0841, -0.1035)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objective value</td>
<td>88.2408</td>
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From the above analysis of the tables, the different non-linear programming algorithms have been carried out and compared. Under the feasible operational region, the steepest ascent method is performed the better solution than the Newton method, even though it has the fast convergence effects.

CONCLUSIONS

With the increasing of quality characteristics and complexity of process, the optimization issues of multi-response problems will become the most important. This paper deals with the multi-responses of different properties that could realistically occur in industry. By using the non-linear programming method of solving the quadratic regression model, we proposed the Newton method and the steepest ascent algorithm for the problem combined means and standard deviations. It presents an iterative procedure for executing the real process optimization involving multiple response variables. The proposed method is first constructed based upon the combined function serving as the model building tool. Then, the direction of steepest ascent direction is applied on the location and dispersion version of the non-linear function, serving as the process improvement tool. Finally, the proposed method is illustrated through the chemical engineering example where simultaneous optimization of several response variables was exercised. The computational results of this numerical example confirm that the proposed method could be a viable tool for the RSM practitioners for performing on-line multi-response process improvement.
RERERENCES


